

Study of a fifth order PDE using symmetries

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Abstract

We study a family of PDEs, which was derived as an approximation of an extended Lotka–Volterra system, from the point of view of symmetries. Also, by performing the self adjoint classification on that family we offer special cases possessing non trivial conservation laws. Using both classifications we justify the particular cases studied in the literature, and we give additional cases that may be of importance.

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1. Introduction

In a recent paper, Zilburg and Rosenau considered the equation

$$u_t = p_1(u^2)_x + h^2 p_3(uu_{xx} + \beta u_x^2)_x + h^4 [q_0 uu_{5x} + q_1 u_x u_{4x} + q_2 u_{2x} u_{3x}] \quad (1)$$

derived by taking an approximation of an extended Lotka–Volterra (ELV) system up to the $\mathcal{O}(h^6)$ order, [9]. Equation (1) contains, among other models, the celebrated $K(2, 2)$ equation¹ — also known as Rosenau–Hyman equation — which is best known for admitting compacton solutions, [8]. Moreover, equation (1) may be capable to describing higher order nonlinear excitations observed in numerical simulations that equations like the KdV fail to replicate, see [7] and referenced therein.

¹ $q_0 = q_1 = q_2 = 0, \beta = 1$ and $h^2 p_3 = 2$

Apart from the Rosenau–Hyman equation, another important special case comes up if we take the choice $q_0 = q_1 = \beta = 0$. Now equation (1) can be rewritten as

$$u_t = \frac{\partial}{\partial x} [bu^2 + 2\kappa uu_{xx} + (u_{xx})^2], \quad (2)$$

where $t \mapsto 2ht/q_2$, $x \mapsto xh$, $b = 2p_1/q_2$ and $\kappa = p_3/q_2$. And if

$$b = \kappa^2, \quad (3)$$

equation (2) becomes

$$u_t = \frac{\partial}{\partial x} [\mathcal{L}u]^2, \quad (4)$$

where \mathcal{L} is the Schrödinger operator with constant potential κ ,

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \kappa.$$

In this paper we shall cast some light into structural properties of equation (1) and improve our knowledge on it, complementing the results obtained in [9]. Along with this desire, we shall investigate the invariance properties of equation (1). First we carry out its Lie group classification and we show how all these special cases found in the literature occur naturally from it. In addition, we give some models of potential interest not found in the literature along with group invariant solutions. This is done in section 2 using the symbolic package SYM for MathematicaTM developed by SD [1].

Once we have the group classification, and hence the Lie Algebra for each possible case, in section 3, we turn our attention to the self adjoint classification of equation (1). Applying the techniques developed in [3, 5], again with the invaluable help of SYM, we unearth special cases possessing non trivial conservation laws. Next, in section 4, we discuss some properties of the solutions of (1) from the conservation laws established in section 3. Our conclusions are presented in section 5.

2. Point symmetries and group invariant solutions

First of all, utilizing the equivalent transformation machinery [4], we can eliminate the parameter h by using the transformation $(x, t, u) \rightarrow (\frac{x}{h}, t, \frac{u}{h})$. Hence, the comment made by the authors in [9] that the equation (1) is quasi continuum — at least from the mathematical standpoint — is not accurate:

up to the cutoff at $\mathcal{O}(h^4)$ the equation do not carry a trace of its discrete origins. Therefore, from this point on we set that $h = 1$. We proceed now with the Lie group classification of equation (1).

2.1. Group classification

By using SYM interactively we arrive to the group classification summarized in table 1:

Case	p_1	p_3	β	q_0	q_1	q_2	Form	Symmetries (span)
0							(1)	$\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$
		general case						
1	0	0	\forall	$\neq 0$	\forall	\forall	$u_t = q_0 u u_{5x} + q_1 u_x u_{4x} + q_2 u_{2x} u_{3x}$	$\mathfrak{X}_4 = x \partial_x + 5u \partial_u$
2	$\neq 0$	0	\forall	0	$\neq 0$	\forall	$u_t = p_1 (u^2)_x + q_1 u_x u_{4x} + q_2 u_{2x} u_{3x}$	$\mathfrak{X}_4 = 2p_1 t \partial_x - \partial_u$
3	0	$\neq 0$	\forall	0	$\neq 0$	0	$u_t = p_3 (u u_{xx} + \beta u_x^2)_x$	$\mathfrak{X}_4 = x \partial_x + 3u \partial_u$
4	0	0	\forall	0	$\neq 0$	\forall	$u_t = q_1 u_x u_{4x} + q_2 u_{2x} u_{3x}$	$\mathfrak{X}_4 = \partial_u,$ $\mathfrak{X}_5 = x \partial_x + 5t \partial_t$
5a	$= \frac{p_3^2}{2q_2}$	$\neq 0$	0	0	0	$\neq 0$	$u_t = \frac{p_3^2}{2q_2} (u^2)_x + p_3 (u u_{xx})_x + q_2 u_{2x} u_{3x}, \kappa > 0$	$\mathfrak{X}_4 = \cos(\sqrt{\kappa} x) \partial_u,$ $\mathfrak{X}_5 = \sin(\sqrt{\kappa} x) \partial_u$
5b	$= \frac{p_3^2}{2q_2}$	$\neq 0$	0	0	0	$\neq 0$	$u_t = \frac{p_3^2}{2q_2} (u^2)_x + p_3 (u u_{xx})_x + q_2 u_{2x} u_{3x}, \kappa < 0$	$\mathfrak{X}_4 = e^{\sqrt{ \kappa x}} \partial_u,$ $\mathfrak{X}_5 = e^{-\sqrt{ \kappa x}} \partial_u$
6	\forall	0	\forall	0	0	0	$u_t = p_1 (u^2)_x$	∞^2

Table 1: Group Classification of (1) where $\mathfrak{X}_1 = \partial_x$, $\mathfrak{X}_2 = \partial_t$, $\mathfrak{X}_3 = t \partial_t - u \partial_u$ and $\kappa = p_3/q_2$.

It is evident that the case studied *ad hoc* in [9], namely the cases 5a and 5b in our classification, are featured by the group classification admitting an exceptional high dimensional Lie algebra in relation with the general case. Besides that, case 4 also admits a 5 dimensional Lie algebra as in the cases 5a and 5b. A fact that suggests that it might be of physical interest too. We continue by providing invariant solutions for these three cases.

²In this case we have a quasi linear PDE that can be easily solved analytically, so there is no need to explicitly present its symmetry algebra here.

2.2. Invariant solutions for case 5

The case 5 corresponds to the family of PDEs³

$$u_t = \pm \frac{1}{2}(u^2)_x + (uu_{xx})_x + u_{2x}u_{3x} \quad (5)$$

that admits a 5-dimensional Lie algebra which is represented by the span of two different sets of five infinitesimal generators depending on the sign of the term $(u^2)_x$, see table 1. Since we are dealing with the same Lie algebra the algebraic part of the analysis that follows is the same and the difference occurs only when we write down the similarity solutions.

As we intend to give the invariant solutions for equation (5) it is advantageous first to construct the optimal set of the corresponding 5-dimensional Lie algebra, see [2, 6] for more details. That is:

\mathfrak{X}_1	\mathfrak{X}_2	\mathfrak{X}_3	\mathfrak{X}_4	\mathfrak{X}_5
$\mathfrak{X}_1 + \mathfrak{X}_2$	$\mathfrak{X}_1 + \lambda \mathfrak{X}_3$	$\mathfrak{X}_4 + \mathfrak{X}_2$		

The invariance under x translations (\mathfrak{X}_1) leads to constant solutions, while the invariance under translations in t (\mathfrak{X}_2) may either provide exponential or sinusoidal solutions of x , depending on the sign. For the remaining symmetries of the optimal set we get the following similarity solutions:

	$+1(\kappa > 0)$	$-1(\kappa < 0)$
$\mathfrak{X}_3 :$	$u = \frac{\phi(x)}{t}$	$u = \frac{\phi(x)}{t}$
$\mathfrak{X}_1 + \mathfrak{X}_2 :$	$u = \phi(x - t)^\dagger$	$u = \phi(x - t)^\dagger$
$\mathfrak{X}_1 + \lambda \mathfrak{X}_3 :$	$u = e^{-\lambda x} \phi(e^{-\lambda x} t)$	$u = e^{-\lambda x} \phi(e^{-\lambda x} t)$
$\mathfrak{X}_4 + \mathfrak{X}_2 :$	$u = \cos(x) t + \phi(x)$	$u = e^x t + \phi(x)$

We note at this point that:

- for the symmetry $\mathfrak{X}_1 + \mathfrak{X}_2$ we get the type of solutions — modulo an equivalence transformation — that Rosenau et. al. studied in [9].

³after applying the equivalence transformation $(x, t, u) \rightarrow (\sqrt{|\kappa|}x, t, q_2\sqrt{|\kappa|^5}u)$.

[†]note the discrete symmetry $(t, u) \rightarrow (-t, -u)$

- To obtain the exact form of the function ϕ we need to substitute each similarity solution to equation (5) and resolve the reduced equation that emerge.

2.3. Invariant solutions for case 4

Now we turn our attention to a special case of equation (1) that is not mentioned in [9], *videlicet*

$$u_t = u_x u_{4x} + q u_{2x} u_{3x}, \quad (6)$$

where $q = \frac{q_2}{q_1}^4$. For this case, the optimal set of the Lie algebra admitted by equation (6) is :

$$\begin{array}{cccc} \mathfrak{X}_1 & \mathfrak{X}_2 & \mathfrak{X}_4 & \mathfrak{X}_5 \\ \mathfrak{X}_3 + \lambda \mathfrak{X}_5 & \mathfrak{X}_1 + \mathfrak{X}_3 & \mathfrak{X}_4 + \mathfrak{X}_5 & \mathfrak{X}_2 + \mathfrak{X}_4 \\ \mathfrak{X}_1 + \mathfrak{X}_2 & \mathfrak{X}_2 + \mathfrak{X}_3 - \frac{1}{5} \mathfrak{X}_5 & & \end{array}$$

Similarly to the previous cases, the invariance under translations in x and t lead to constant solutions and implicit solutions involving Hypergeometric functions respectively. The third symmetry of the optimal set, \mathfrak{X}_4 , only denotes that the function itself do not appear in (6) and as such gives no invariant solutions⁵. For the remaining symmetries of the optimal set we have the following similarity solutions:

$$\begin{aligned} \mathfrak{X}_5 : u &= \phi\left(\frac{x^5}{t}\right) \\ \mathfrak{X}_3 + \lambda \mathfrak{X}_5 : u &= \begin{cases} x^{-\frac{1}{\lambda}} \phi\left(\frac{x^{5+\frac{1}{\lambda}}}{t}\right), & \lambda \neq 0 \\ \frac{\phi(x)}{t}, & \lambda = 0 \end{cases} \\ \mathfrak{X}_1 + \mathfrak{X}_3 : u &= e^{-x} \phi(e^{-x} t) \\ \mathfrak{X}_4 + \mathfrak{X}_5 : u &= \phi\left(\frac{x^5}{t}\right) + \ln x \end{aligned}$$

⁴after applying the equivalence transformation $t \rightarrow q_1 t$

⁵It affirms the fact that if u is a solution so is $u + c$ where c a constant.

$$\begin{aligned}
\mathfrak{X}_2 + \lambda \mathfrak{X}_4 : u &= \phi(x) + t \\
\mathfrak{X}_1 + \lambda \mathfrak{X}_2 : u &= \phi(x - t) \\
\mathfrak{X}_2 + \mathfrak{X}_3 - \frac{1}{5} \mathfrak{X}_5 : u &= x^5 \phi(x^5 e^t)
\end{aligned}$$

3. Self adjoint classification and conservation laws derived from point symmetries

In addition to the group classification we performed the self adjoint classification of the PDE, that is, to reveal the special cases that are (strictly, quasi and nonlinearly) self adjoint. Ibragimov proposed in [3] a procedure that utilize that property to construct conservation laws by introducing the concept of a *formal Lagrangian* \mathcal{L} . A formal Lagrangian is nothing more than the differential equation multiplied by a new dependent variable $v(x, y)$, *viz.*

$$\begin{aligned}
\mathcal{L} = v \Delta = \\
v(u_t - p_1(u^2)_x - h^2 p_3(uu_{xx} + \beta u_x^2)_x - h^4 [q_0 u u_{5x} + q_1 u_x u_{4x} + q_2 u_{2x} u_{3x}]).
\end{aligned}$$

When

$$\frac{\delta}{\delta u}(\mathcal{L}) = \lambda \Delta,$$

where $\frac{\delta}{\delta u}$ the Euler operator, our equation is self adjoint. It is clear that the self adjointness depends on the choice of the dependent variable v :

- if $v = u$ the equation is *strictly* self adjoint;
- if $v = \Phi(u)$ the equation is *quasi* self adjoint;
- if $v = \Phi(x, t, u)$ the equation is *nonlinearly* self adjoint, and finally
- if $v = \Phi(x, t, u, u_x, u_{xx}, \dots)$ the equation is *generalized nonlinearly* self adjoint.

The fact that a differential equation is self adjoint means that its symmetries are also variational symmetries. Therefore, we can use the Noether theorem, expressed as an operator identity,

$$\mathfrak{X} + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathcal{N}^i,$$

where $\mathfrak{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ the chosen symmetry appropriately prolonged and $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, to obtain the corresponding conserved vector $\mathcal{N}(\mathcal{L})$.

By solving the identity for \mathcal{N} we obtain the formula:

$$\mathcal{N}^i \mathcal{L} = \xi^i \mathcal{L} + \sum_{i_1 + \dots + i_n = 0}^{\infty} D_{i_1} \dots D_{i_n} (W^\alpha) \frac{\delta^* \mathcal{L}}{\delta^* u_{i_1 \dots i_n}^\alpha},$$

with

$$\frac{\delta^*}{\delta^* u_{i_1 \dots i_n}^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=j_1 + \dots + j_n = 1}^{\infty} (-1)^s \frac{\binom{s}{j_1, \dots, j_n}}{\binom{s+i_1+\dots+i_n}{i_1+j_1, \dots, i_n+j_n}} D_{j_1} \dots D_{j_n} \frac{\partial}{\partial u_{(i_1+j_1) \dots (i_n+j_n)}^\alpha},$$

where $\binom{N}{i_1, i_2, \dots, i_r} = \frac{N!}{i_1! i_2! \dots i_r!}$, $N = i_1 + i_2 + \dots + i_r$ is the multinomial and $i_j \geq 0$ denotes the order of the derivative for the j^{th} independent variable. For a more detailed survey on the aforementioned concepts, fully integrated in SYM, see also [3, 5].

3.1. The strictly self adjoint cases

The cases $q_2 = 5(q_1 - 2q_0)$, $p_3 = 0$ and $q_2 = 5(q_1 - 2q_0)$, $\beta = \frac{1}{2}$, namely the PDEs

$$\begin{aligned} u_t = & p_1(u^2)_x + [q_0 u u_{5x} + q_1 u_x u_{4x} + 5(q_1 - 2q_0) u_{2x} u_{3x}] = \\ & p_1(u^2)_x + \left(q_0 u u_{4x} + (q_1 - q_0) u_x u_{3x} + (2q_1 - \frac{9}{2} q_0) u_{xx}^2 \right)_x \end{aligned} \quad (7)$$

and

$$\begin{aligned} u_t = & p_1(u^2)_x + p_3(u u_{xx} + \frac{1}{2} u_x^2)_x + [q_0 u u_{5x} + q_1 u_x u_{4x} + 5(q_1 - 2q_0) u_{2x} u_{3x}] = \\ & p_1(u^2)_x + p_3(u u_{xx} + \frac{1}{2} u_x^2)_x + \\ & \left(q_0 u u_{4x} + (q_1 - q_0) u_x u_{3x} + (2q_1 - \frac{9}{2} q_0) u_{xx}^2 \right)_x, \end{aligned} \quad (8)$$

are strictly self adjoint. Observe that both cases can be written as conservation laws, the reason will be revealed, and proved, in the following section.

Case	p_1	p_3	β	q_0	q_1	q_2	Φ
1	\forall	0	\forall	0	0	0	$\Phi(u)$
2	\forall	0	\forall	$\neq 0$	$\neq q_0$	$5(q_1 - q_0)$	$c_1 + c_2 \ln u$
3	\forall	$\neq 0$	0	\forall	q_0	0	$c_1 + c_2 \ln u$
4	\forall	0	\forall	$\neq 0$	$\neq q_0$	0	$c_1 + c_2 u^{\frac{q_1}{q_0} - 1}$
5	\forall	$\neq 0$	$\neq 0$	\forall	$(1 + 2\beta)q_0$	0	$c_1 + c_2 u^{2\beta}$
6	\forall	0	\forall	0	$\neq 0$	$5q_1$	$c_1 + c_2 u + c_3 u^2$
7	\forall	0	\forall	$\neq 0$	\forall	$5(q_1 - 3q_0)$	$c_1 + c_2 u^2$
8	\forall	$\neq 0$	1	\forall	\forall	$5(q_1 - 3q_0)$	$c_1 + c_2 u^2$
9	\forall	0	\forall	0	\forall	$\neq 5q_1$	c_1
10	\forall	0	\forall	$\neq 0$	q_0	0	c_1
11	\forall	\neq	0	\forall	\forall	$\neq 0$	c_1
12	\forall	0	\forall	$\neq 0$	\forall	$\neq 0$	c_1
						and	
						$\neq 5(q_1 - q_0)$	
						and	
13	\forall	0	\forall	$\neq 0$	\forall	$\neq 5(q_1 - 2q_0)$	$c_1 + c_2 u$
						and	
14	\forall	$\neq 0$	$\frac{1}{2}$	\forall	\forall	$\neq 5(q_1 - 3q_0)$	$c_1 + c_2 u$
						$5(q_1 - 2q_0)$	

Table 2: Quasi self adjoint classification of equation (1), $\Phi(u)$ is an arbitrary function of u

3.2. The quasi self adjoint cases

We turn now to the quasi self adjoint case. The result of this kind of classification is gathered in table 2.

We observe that:

- In any case v can be a constant. This means that the equation (1) can be written in the form $u_t = (\cdot)_x$. Verily,

$$u_t = p_1(u^2)_x + (h^2 u(h^2 q_0 u_{xx} + p_3 u)_{xx})_x + \frac{(q_0 - q_1 + q_2)}{2} h^4 (u_{xx}^2)_x + h^2 (u_x(\beta p_3 u - h^2(q_0 - q_1)u_{xx})_x)_x.$$

- The special case briefly discussed in [9],

$$u_t = p_1(u^2)_x + p_3(uu_{xx})_x + q_0(uu_{4x})_x,$$

corresponds to our case 3 in table 2. And apart from the obvious conservation law — using either the symmetry \mathfrak{X}_2 or \mathfrak{X}_2 and $\Phi = 1$ — it has also another one. Namely,

$$(\log(u)u)_t - \left(\frac{1}{2} (2(\log(u) + 1)u (p_3 u_{xx} + q_0 u_{xxxx}) + p_1(2 \log(u) + 1)u^2 - u_x (p_3 u_x + 2q_0 u_{xxx}) + q_0 u_{xx}^2) \right)_x = 0.$$

obtained with the symmetry \mathfrak{X}_3 and $\Phi = \log u$.

- The last two cases correspond to the strictly self adjoint ones and the existence of the constant explains the way we have written them in the previous section.

3.3. The generalized nonlinearly self adjoint cases

Beyond the special cases we illustrated so far equation (1) possesses a case that has the rare property to be generalized nonlinearly self adjoint. That happens when $p_1 = 0, \beta = -\frac{1}{4}$ and $q_1 = \frac{3}{2}q_0 \neq 0$, that is

$$u_t = \frac{1}{4} (4u (p_3 u_{xx} + q_0 u_{xxxx}) - p_3 u_x^2 - (q_0 - 2q_2)u_{xx}^2 + 2q_0 u_{xxx} u_x)_x.$$

This case admits the substitution $\Phi = u_{xx}$ which in combination with the symmetry \mathfrak{X}_3 yields the non trivial conservation law:

$$\begin{aligned} (u_x^2)_t + \left(u \left(p_3 u_{xx}^2 - q_0 u_{xxx}^2 + 2q_0 u_{xx} u_{xxxx} \right) + \right. \\ \left. q_0 u_x u_{xx} u_{xxx} + \frac{2q_2 - q_0}{3} u_{xx}^3 - 2u_x u_t \right)_x = 0. \end{aligned}$$

4. Remarks on the constants of motion

Here we present some facts regarding the conservation laws established in the previous section.

1. Since equation (1) is itself a conservation law, this implies that the quantity

$$\mathcal{H}_0[u] = \int_{-\infty}^{+\infty} u \, dx \quad (9)$$

is a constant of motion. Particularly, if u is a non-negative function, (9) implies on the conservation of the $L^1(\mathbb{R})$ -norm of the solutions of (1) rapidly decaying to 0, jointly with its derivatives, at the infinity.

2. In the strictly self adjoint cases (cases 6, 13 and 14 in Table 2) we have the conserved quantity u^2 , which implies that

$$\mathcal{H}_1[u] = \int_{-\infty}^{+\infty} u^2 \, dx \quad (10)$$

is also a constant of motion, which mathematically corresponds to the existence of square integrable solutions of (1).

3. Cases 2 and 3 in Table 2 have a constant of motion of the logarithmic type:

$$\mathcal{H}_2[u] = \int_{-\infty}^{+\infty} u \ln |u| \, dx. \quad (11)$$

4. For the case $p_1 = 0, \beta = -\frac{1}{4}$ and $q_1 = \frac{3}{2}q_0 \neq 0$, in addition to (9), we have the constant of motion

$$\mathcal{H}_3 = \int_{-\infty}^{+\infty} u_x^2 \, dx. \quad (12)$$

In addition, if $q_2 = -5q_0/2$ the PDE is also quasi self adjoint, falling into the case 13, hence admitting also (10) as a constant of motion

on the (rapidly decaying) solutions of (1). Combining (10) and (12), we obtain the conservation of the $H^1(\mathbb{R})$ -norm of the solutions of (1) satisfying these constraints.

5. Cases 4, 5, 6, 7 and 8 have the constant of motion

$$\mathcal{H}_4[u] = \int_{-\infty}^{+\infty} u^\sigma dx, \quad \sigma \neq 0 \quad (13)$$

where $\sigma = q_1/q_2$ (case 4), $\sigma = 2\beta + 1$ (case 5) and $\sigma = 3$ for the remaining cases.

5. Conclusion

In the present work we showed how symmetries can help in a systematic and thorough study of a family of nonlinear PDEs of the fifth order. By its group classification, and then, its self adjoint classification we were able not only to retrieve the cases that Zilburg and Rosenau studied *ad hoc* but also to give additional cases possessing non trivial, and not obvious, conservation laws. Also we proved that equation (1) can be written in the form $u_t = (\cdot)_x$ for any value of the parameters. A bit of information that greatly helps the study of equation (1) that Zilburg and Rosenau performed for only two special cases in [9] commenting that “in its full glory appears to be well beyond our ability to analyze it”. Actually, this very comment was our main motivation for studying this class of PDEs by looking into their structure — their *DNA* in a matter of speaking — its symmetries. From them, we were able to construct solutions and conservation laws of (1), another fact that shows the usefulness of the symmetry analysis we performed. Furthermore, we uncovered a very interested case, namely the case $p_1 = 0, \beta = -\frac{1}{4}$ and $q_1 = \frac{3}{2}q_0 \neq 0$, where we have a generalized nonlinear self adjoint PDE which yields a conservation law with a higher order characteristic.

In a future work we will utilize the symmetry machinery in order to study systematically higher order expansion cutoffs of the ELV system.

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